*J. Appl. Maths Mechs* Vol. 57, No. 3, pp. 439–445, 1993
Printed in Great Britain.

0021-8928/93 $24.00+.00


# AN OPTIMIZATION MODEL OF THE STEFAN PROBLEM†

V. A. KALISTRATOV and S. P. OKHEZIN

Yekaterinburg

(*Received* 1 *April* 1992)

It was shown in [1, 2, 4], when approximating the parabolic Stefan problem by a family of optimal-control problems in which the controlling parameter is the form of the region in which the state-temperature function of the liquid phase is defined, that the classical solution of the Stefan problem is the limit of the solutions of the corresponding approximation problems in the metrics of approximate functional spaces. The structure of the optimal-control approximation problems which model the Stefan problem is investigated, the necessary conditions for these problems to be solvable are obtained, and a proof of the convergence of the proposed approximate methods of solving them is given.

## 1. FORMULATION OF THE PROBLEM

THE FOLLOWING model of the one-dimensional frontal and single-phase Stefan problem is considered

$$y^{\cdot}(t,x)=y''(t,x),\quad 0<t\leqslant T,\quad 0<x<u(t) \tag{1.1}$$

$$y(0,x)=\varphi_0(x),\quad 0\leqslant x\leqslant u_0 \tag{1.2}$$

$$y(t,0)=0,\quad y(t,u(t))=0,\quad 0\leqslant t\leqslant T \tag{1.3}$$

$$y'(t,u(t))=-ku^{\cdot}(t),\quad 0<t\leqslant T \tag{1.4}$$

$$u(0)=u_0 \tag{1.5}$$

Here $y(t,x)$ is the temperature of the liquid phase at the point $x$ at the instant of time $t$. The function $\varphi_0(\cdot)$ describes the initial temperature distribution in the liquid phase, and $u(t)$ is the change in the form of the region occupied by the liquid phase. Condition (1.4) is the mathematical form of the heat-balance equation at the interface between the liquid and the solid phases. It is assumed in the model that the temperature of the solid phase is equal to the temperature of the phase transition and is zero. The dot denotes differentiation with respect to $t$, and the prime denotes differentiation with respect to $x$.

Problem (1.1)–(1.5), as in [1, 2], becomes an optimal-control problem of special structure

$$y_\varepsilon^{\cdot}(t,x)-y_\varepsilon''(t,x)+\varepsilon^{-1}U_\varepsilon(t,x;u(\cdot))y_\varepsilon(t,x)=0,\quad 0<t\leqslant T,\quad 0\leqslant x<X$$
$$y_\varepsilon(0,x)=\begin{cases}\varphi_0(x), & x\in[0,u_0]\\ 0, & x\in(u_0,X]\end{cases}$$

†*Prikl. Mat. Mekh.* Vol. 57, No. 3, pp. 34–40, 1993.

$$y_\varepsilon(t,0)=y_\varepsilon(t,X)=0,\quad 0\leq t\leq T$$
$$u^{\cdot}(t)=\upsilon(t),\quad u(0)=u_0 \tag{1.6}$$

Here $\varepsilon>0$ is a numerical parameter, and the function $\upsilon(t)$ plays the role of the control, and

$$U_\varepsilon(t,x;u(\cdot))=\begin{cases}0, & 0\leq x\leq u(t)\\ \exp[-(x-u(t)-\varepsilon)^2(x-u(t))^{-2}] & \\ u(t)<x\leq u(t)+\varepsilon & \\ 1, & u(t)+\varepsilon<x\leq X\end{cases} \tag{1.7}$$

In the solution of problem (1.6) the quality functional, produced by condition (1.4), is specified

$$J(\varepsilon,\upsilon(\cdot))=\int_0^T[y'_\varepsilon(t,u(t))+k\upsilon(t)]^2\,dt \tag{1.8}$$

In problem (1.1)–(1.5) using the initial function $\varphi_0(\cdot)$ we can estimate a priori the value of the derivative $y'(t,x)$ on the front $x=u(t)$

$$0\leq y'(t,u(t))\leq V=k^{-1}\max\{|\varphi_0(x)(x-u_0)^{-1}|,\quad x\in[0,u_0]\} \tag{1.9}$$

The number $x$ is chosen in such a way that when $t\in(0,T)$ we have $u(t)\in(0,X)$, for example $X=VT+u_0$.

We choose as the set of permissible controls $\upsilon(t)$ the set

$$V[0,T]=\{\upsilon(\cdot)|\upsilon(t)\in[0,V],\quad\forall\, t\in[0,T]\}$$

We will consider the following fundamental approximation problem.

For a specified $\varepsilon>0$ it is required to obtain the control $\upsilon^*_\varepsilon(\cdot)\in V[0,T]$ such that

$$J(\varepsilon,\upsilon^*_\varepsilon(\cdot))=\inf\{J(\varepsilon,\upsilon(\cdot))\mid\upsilon(\cdot)\in V[0,T]\}$$

We will formulate the main results [2] as they relate to the approximation problem.

*Theorem* 1.1. The functional $J(\varepsilon,\upsilon(\cdot))$ is semi-continuous from below with respect to a change in $\upsilon(\cdot)$ in a weak topology of the space $L_2(0,T)$.

*Theorem* 1.2. In the set $V[0,T]$, the functional $J(\varepsilon,\upsilon(\cdot))$ reaches an accurate lower limit.

*Theorem* 1.3. For any sequence $u_{\varepsilon_n}(\cdot)\to u_*(\cdot)$ in $C[0,T]$ as $\varepsilon_n\to 0$ convergence occurs in $L_2(0,T)$

$$y'_{\varepsilon_n}(t,u_{\varepsilon_n}(t);u_{\varepsilon_n}(\cdot))\to y'(t,u_*(t);u_*(\cdot)) \tag{1.10}$$

Here $y_\varepsilon(t,x;(\cdot))$ is the solution of problem (1.6) corresponding to the chosen function $u(\cdot)$, while $y(t,x;u(\cdot))$ is the solution of the heat-conduction equation (1.1)–(1.3) in a region that is non-cylindrical with respect to the variable $t$

$$Q(u)=\{(t,x)|\,0<x<u(t),\ 0<t<T\}$$

*Theorem* 1.4. As $\varepsilon\to 0$ we have the convergence

$$u_\varepsilon(\cdot)\to u_0(\cdot)\quad(u^{\cdot}_\varepsilon(t)=\upsilon_\varepsilon(t))\ \text{in}\ C[0,T]$$
$$y_\varepsilon(\cdot,\cdot;u_\varepsilon(\cdot))\to y_0(\cdot,\cdot)\ \text{in}\ L_2(Q(u_0)) \tag{1.11}$$

Here $\upsilon_\varepsilon(\cdot)$ is a function which makes the functional $J(\varepsilon, \upsilon(\cdot))$ a minimum in the set $V[0, T]$, $y_\varepsilon(\cdot, \cdot; u_\varepsilon(\cdot))$ is the corresponding solution of problem (1.6), and $u(\cdot)$ and $y_0(\cdot, \cdot)$ is the classical solution of the Stefan problem (1.1)–(1.5).

It follows from Theorem 1.4 that any sequence $\{u^*_{\varepsilon_n}(\cdot), y^*_{\varepsilon_n}(\cdot, \cdot)\}$, which solves the approximation problem, converges to $\{u_0(\cdot), y_0(\cdot, \cdot)\}$, and for sufficiently small $\varepsilon_n > 0$ can be chosen as the approximate solution of the Stefan problem.

Hence, the problem arises of constructing a set of functions $\upsilon^*_\varepsilon(\cdot)$, which solves the approximation problem for fixed $\varepsilon > 0$.

## 2. MAIN RESULTS

The approximation system (1.6) is a standard Dirichlet boundary-value problem for a parabolic-type equation defined in the cylindrical region $D = (0, T) \times (0, X)$. We will derive the necessary conditions which any controlled $\upsilon^*_\varepsilon(\cdot)$, which solves the approximation problem, must satisfy.

The following boundary-value problem will be called a conjugate system

$$w^{\cdot}_\varepsilon(t,x) + w''_\varepsilon(t,x) - \varepsilon^{-1} U_\varepsilon(t,x;u(\cdot)) w_\varepsilon(t,x) =$$

$$= 2(y'_\varepsilon(t,u(t);u(\cdot)) + k\upsilon(t))\delta'(x - u(t)), \quad (t,x) \in D \tag{2.1}$$

$$w_\varepsilon(T, x) = 0, \quad 0 \leqslant x \leqslant X \tag{2.2}$$

$$w_\varepsilon(t, 0) = w_\varepsilon(t, X) = 0, \quad 0 \leqslant t \leqslant T \tag{2.3}$$

$$u^{\cdot}(t) = \upsilon(t), \quad u(0) = u_0 \tag{2.4}$$

Here $\delta'$ is the derivative in space of the distributions $D^*(0, X)$ of the Dirac $\delta$-distribution, while $y_\varepsilon(\cdot, \cdot; u(\cdot))$ is the classical solution of boundary-value problem (1.6) corresponding to the function $u(\cdot)$.

Any distribution $w_\varepsilon \in D^*(0, T; H^*)$ which satisfies Eq. (2.1) with final condition (2.2) and boundary condition (2.3), in the sense of the theory of distributions, will be called a solution of the conjugate system.

*Notes.* 1. We will denote by $H^*$ the space conjugate to $H = H^1_0(0, X) \cap H^2(0, X)$.

2. Using the technique of the expansion of the elements of the space $H^*$ in series in a specially chosen basis in the space $H$, it can be shown that the distribution $w_\varepsilon$ is generated by an element of the space $L_2(0, T; H^*)$ and, moreover, this element has a representative which is a function with values in the space $H^*$ continuous in $[0, T]$. In view of this, the final condition $w_\varepsilon(T, x) = 0$ has a meaning in this space.

3. We will assume that the function $\varphi_0(\cdot)$ in condition (1.2) is such that the extension to zero outside $[0, u_0]$ gives an element of the space $H^3(0, X)$.

*Theorem* 2.1. The conjugate system has a unique solution which is an element of the space $L_2(0, T; H^*)$.

We will indicate the main elements of the proof of this theorem. In the space $H = H^1_0(0, X) \cap H^2(0, X)$ we distinguish a basis consisting of the eigenfunctions of the problem

$$-\omega''_j(x) = \lambda_j \omega_j(x), \quad x \in (0, X)$$

$$\omega_j(0) = \omega_j(X) = 0 \tag{2.5}$$

The following equivalent integral equation for the function $w_\varepsilon$, which generates the distribution $w_\varepsilon$, corresponds to (2.1) in the space $D^*(0, T; H^*)$

$$z_\varepsilon(s,x) = -\sum_{j=1}^{\infty}\left(\int_0^s \exp(-\lambda_j(s-\tau))f(\tau)\omega_j'(q(\tau))d\tau)\omega_j(x) - \right.$$

$$- \sum_{j=1}^{\infty}\varepsilon^{-1}\left(\int_0^s \exp(-\lambda_j(s-\tau)\right)\langle U_\varepsilon(T-\tau,\cdot;q(\cdot))z_\varepsilon(\tau,\cdot),\omega_j(\cdot)\rangle_{L_2(0,X)}d\tau\Bigg)\omega_j(x) \tag{2.6}$$

$$z_\varepsilon(s,x) = w_\varepsilon(T-s,x), \quad q(\tau) = u(T-\tau)$$

$$f(\tau) = 2(y_\varepsilon'(T-\tau,q(\tau);q(\cdot)) + kv(T-\tau))$$

The integral equation (2.6) can be written in the following operator form

$$z_\varepsilon(\cdot,\cdot) = P(f(\tau)\delta'(x-q(\tau)))(\cdot,\cdot) + Tz_\varepsilon(\cdot,\cdot) \tag{2.7}$$

When $s \in [0, S_0]$, where $S_0$ is a fairly small positive number, the operator (2.7) is compressive in the space $L_2(0, S_0; H^*)$. By splitting the section $[0, T]$ into a finite number of intervals of length not greater than $S_0$, it can be shown that Eq. (2.1) has a unique solution in the space $L_2(0, T; H^*)$. Applying standard constructions, connected with Lagrange's method of eliminating bonds (limitations), following, for example, [4–6], we obtain the following representation for the gradient of the functional $J(\varepsilon, \upsilon(\cdot))$ in the space $L_2(0, T)$

$$\begin{aligned} \nabla J(\varepsilon,\upsilon(\cdot))(\tau) &= 2k(y_\varepsilon'(\tau,u(\tau);u(\cdot)) + k\upsilon(\tau)) + \\ &+2\int_0^T (y_\varepsilon'(t,u(t);u(\cdot)) + k\upsilon(t))y_\varepsilon''(t,u(t);u(\cdot))K(t,\tau)dt - \\ &-\varepsilon^{-1}\iint_D y_\varepsilon(t,x;u(\cdot))w_\varepsilon(t,x;u(\cdot))U_\varepsilon'(t,x;u(\cdot))K(t,\tau)dxdt \\ K(t,\tau) &= \begin{cases} 0, & \tau > t \\ 1, & \tau \leq t \end{cases} \end{aligned} \tag{2.8}$$

Here $y_\varepsilon(t, x; u(\cdot))$ is the solution of problem (1.6) and $w_\varepsilon(t, x; u(\cdot))$ is the solution of the conjugate system.

Taking into account the theorems on traces in Sobolev spaces [4, 5] it can be shown that $\nabla J(\varepsilon, \upsilon(\cdot))(\cdot) \in L_2(0, T)$. Hence, the necessary condition for an extremum , which is satisfied by any element $\upsilon_\varepsilon^*(\cdot)$, which solves the approximation problem, is formulated in the following theorem [6].

*Theorem* 2.2. Suppose $\upsilon_\varepsilon^*(\cdot)$ is the solution of the approximation problem. Then the following condition is necessarily satisfied

$$\langle \nabla J(\varepsilon,\upsilon_\varepsilon^*(\cdot))(\cdot), \upsilon(\cdot) - \upsilon_\varepsilon^*(\cdot)\rangle_{L_2(0,T)} \geq 0, \quad \forall \upsilon(\cdot) \in V[0,T] \tag{2.9}$$

We will denote by $\Pi$ the operator of projection in the space $L_2(0, T)$ onto a convex closed set $V[0, T]$. The necessary condition (2.9) can be formulated as follows [6].

*Theorem* 2.3. Suppose $\upsilon_\varepsilon^*(\cdot)$ is the solution of the approximation problem. Then, for all $\alpha > 0$ the following equation holds

$$\upsilon_\varepsilon^*(\cdot) = \Pi(\upsilon_\varepsilon^*(\cdot) - \alpha\nabla J(\varepsilon,\upsilon_\varepsilon^*(\cdot))(\cdot)) \tag{2.10}$$

Theorem 2.3 serves as the basis for approximate algorithms which enable us to construct sequences which approximate the extremal elements $\upsilon_\varepsilon^*(\cdot)$ in the metric of the space $L_2(0, T)$.

The well-known results in [6], which relate to such algorithms, for example, the method of gradient projection, rest on the following fact.

*Theorem* 2.4. The gradient of the quality functional $J(\varepsilon, \upsilon(\cdot))$ satisfies the Lipschitz condition on the set $V[0, T]$, i.e.

$$\exists M > 0 \ \forall \upsilon_1(\cdot), \quad \upsilon_2(\cdot) \in V[0, T]:$$

$$\|\nabla J(\varepsilon, \upsilon_1(\cdot))(\cdot) - \nabla J(\varepsilon, \upsilon_2(\cdot))(\cdot)\|_{L_2(0,T)} \leqslant M\|\upsilon_1(\cdot) - \upsilon_2(\cdot)\|_{L_2(0,T)}$$

Tracing the structure of the terms in the formula for the gradient of the functional (2.8), we conclude that to prove the theorem it is sufficient to show that the Lipschitz condition is satisfied for the following mappings

$$\upsilon(\cdot) \to y'_\varepsilon(\cdot, u(\cdot); u(\cdot)); \upsilon(\cdot) \to y''_\varepsilon(\cdot, u(\cdot); u(\cdot))$$

$$\upsilon(\cdot) \to y_\varepsilon(\cdot, \cdot; u(\cdot)); \quad \upsilon(\cdot) \to U'_\varepsilon(\cdot, \cdot; u(\cdot))$$

$$\upsilon(\cdot) \to U_\varepsilon(\cdot, \cdot; u(\cdot)); \quad \upsilon(\cdot) \to w_\varepsilon(\cdot, \cdot; u(\cdot)) \tag{2.11}$$

where the functions $u(\cdot)$ and $\upsilon(\cdot)$ are connected by the penultimate relation of (1.6).

The satisfaction of the Lipschitz condition for the first five mappings of (2.11) is a consequence of the linearity of the boundary-value problem (1.6) with respect to $y_\varepsilon$ and the properties of the penalty function $U_\varepsilon$. Taking (2.7) into account, to prove that the Lipschitz condition is satisfied for the last mapping of (2.11), it is sufficient to show this for the mapping

$$\upsilon(\cdot) \to P(f(\tau)\delta'(x - q(\tau))(\cdot, \cdot)$$

By the definition of the function $f(\cdot)$ the mapping $\upsilon(\cdot) \to f(\cdot)$ satisfies the Lipschitz condition with constant $C$.

Taking into account the definition of the operator $P$ (2.7), we obtain the limit

$$\|P(f_1(\tau)\delta'(x - q_1(\tau))) - P(f_2(\tau)\delta'(x - q_2(\tau)))\|^2_{L_2(0,T;H^*)} =$$

$$= \sum_{j=1}^{\infty} \lambda_j^{-2} \int_0^T \left( \int_0^t \exp(-\lambda_j(t-\tau))(f_2(\tau)\omega'_j(q_2(\tau)) - f_1(\tau)\omega'_j(q_1(\tau)))d\tau \right)^2 dt \leqslant$$

$$\leqslant \sum_{j=1}^{\infty} \lambda_j^{-2} \int_0^T \left\{ 4X^{-1}\lambda_j^2 \left( \int_0^t \exp(-\lambda_j(t-\tau))|f_2(\tau)||q_2(\tau) - q_1(\tau)|d\tau \right)^2 + \right.$$

$$\left. +4X^{-1} \left( \int_0^t \exp(-\lambda_j(t-\tau))\sqrt{\lambda_j}\,|f_2(\tau) - f_1(\tau)|d\tau \right)^2 \right\} dt \leqslant AI + BI$$

$$A = 4T^2KX^{-1} \sum_{j=1}^{\infty} \lambda_j^{-1}, \quad B = 4C^2X^{-1} \sum_{j=1}^{\infty} \lambda_j^{-2}, \quad I = \int_0^T |\upsilon_1(\tau) - \upsilon_2(\tau)|^2 d\tau$$

Here we have taken into account the fact that $\omega_j(\cdot)$, $\lambda_j$ solve problem (2.5), and in view of the uniformity of the first equation of (1.6) and the smoothness of the function $U_\varepsilon$, the quantity $f_2(\tau)$ is estimated by a certain constant $K$ uniformly with respect to $\tau \in [0, T]$ and $\upsilon(\cdot) \in V[0, T]$

$$|f_2(\tau)| \leqslant K, \quad \forall \tau \in [0, T], \quad \forall \upsilon(\cdot) \in V[0, T]$$

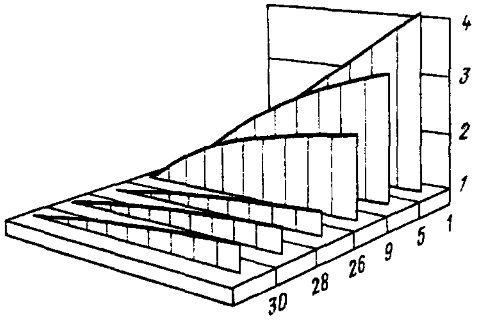


FIG. 1.

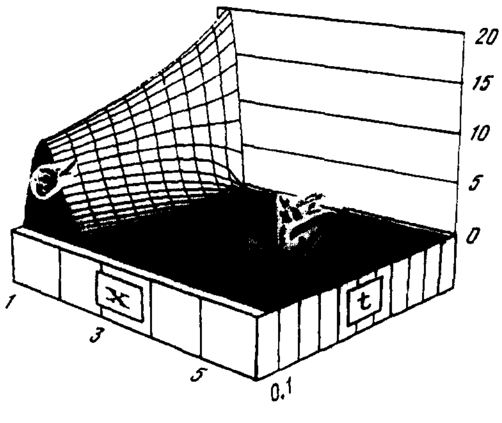


FIG. 2.

Choosing the number $M_1 > 0$ so that $M_1^2 = \max\{A, B\}$, we obtain that the Lipschitz condition is satisfied for the last mapping of (2.11), and Theorem 2.4 is proved.

*Example.* Taking Theorem 2.4 into account for the numerical solution of the approximation problem we used the gradient projection method [6].

The initial function $\varphi(x)$ was approximated by the function

$$\psi(x) = \begin{cases} A_0 \sin \pi u_0^{-1} x, & 0 \leqslant x < u_0 \\ 0, & u_0 \leqslant x \leqslant X \end{cases}$$

which in the neighbourhood of the point $u_0$ was smoothed in the appropriate way. The number $X$ was calculated from the formula $X = [TV] + 1 + h$, where $[\cdot]$ is the integer part of the number, $V$ was chosen from (1.9), and $h$ is the step of the grid with respect to the variable $x$.

The function $U_\varepsilon$ was approximated by a fifth-order spline, which ensured that the solutions of the systems considered were sufficiently smooth. The distribution $\delta'(x-u(t))$ was approximated by a smooth regularization of the functions of the form

$$-\pi^{1/2} N^3 2(x-u(t)) \exp(-N^2 (x-u(t))^2)$$

for appropriate values of $N$. To calculate $y_\varepsilon$ and $w_\varepsilon$ we made use of implicit difference schemes which were solved by the pivotal condensation method. The calculations were carried out for the following values of the parameters

$$T = 0,1; \quad u_0 = 1; \quad A_0 = 20; \quad \varepsilon = 0,1; \quad N = 20; \quad k = 2$$

the step with respect to $t$, $\tau = 0.01$, and the step with respect to $x$, $h = 0.05$.

Figure 1 shows graphs of the elements of the iterative sequence $u^{(s)}(t)$, which approximates the actual front $u_0(t)$. Figure 2 shows the corresponding temperature surface $y_\varepsilon(t, u; u_0(\cdot))$. Here, the corresponding extremal value of the quality functional $J(\varepsilon, v(\cdot))$ was equal to 0.001601.

The calculations were carried out for different values of the system parameters and show that the proposed method gives a computer approximation of the Stefan problem.

## REFERENCES

1. OKHEZIN S. P., An approximation in the problem of controlling the form of a region for a parabolic system. *Prikl. Mat. Mekh.* **54**, 3, 361–365, 1990.
2. OKHEZIN S. P., A mathematical Stefan problem. *Diff. Urav.* **27**, 6, 1042–1048, 1991.

3. MEIRMANOV A. M., *The Stefan Problem.* Nauka, Novosibirsk, 1986.
4. LIONS J.-L. and MADJENES E., *Non-uniform Boundary-value Problems and their Applications.* Mir, Moscow, 1971.
5. ADAMS R. A., *Sobolev Spaces*. Academic Press, New York, 1975.
6. VASIL'YEV F. P., *Methods of Solving Extremal Problems.* Nauka, Moscow, 1981.

*Translated by* R.C.G.